Chapter 1 Solution Techniques in Electrostatics and Magnetostatics

1.1 MAXWELL'S EQUATIONS

Modern electromagnetism is based on a set of four fundamental relations known as Maxwell's equations**.** They are given below in Integral and Differential forms:

	Integral form	Differential form
(1) Gauss's law: (for electric fields)	$\oint_S \vec{\mathbf{D}} \cdot d\vec{s} = \mathbf{Q} = \int_{vol} \rho_v dv$	$\nabla \cdot \vec{\mathbf{D}} = \rho_v$
(2) Ampere's law:	$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \mathbf{I_c} + \mathbf{I_d} = \mathbf{I}$	$\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial \vec{D}}{\partial t}$

($\mathbf{I}_c$ = conduction current $\mathbf{I_d}$ = displacement current= $\int_s \frac{\partial \vec{D}}{\partial t}\mathbf{.}dS$ **)**

	Integral form	Differential form
(3) Faraday's law:	$\oint_{\mathbf{C}} \vec{\mathbf{E}} \cdot d\vec{\ell} = - \int_{\mathbf{S}} \frac{\partial \vec{\mathbf{B}}}{\partial \mathbf{t}} \cdot d\vec{s}$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
(4) Gauss's law: (for magnetic fields)	$\oint_S \vec{\mathbf{B}} \cdot d\vec{s} = 0$	$\nabla \cdot \vec{\mathbf{B}} = 0$

For harmonic variation of fields, $\frac{\partial}{\partial t} = j\omega$ and the Maxwell's equation will take the form given below:

	Integral form	Differential form
(1) Gauss's law: (for electric fields)	$\oint_S \vec{\mathbf{D}} \cdot d\vec{s} = \mathbf{Q}$	$\nabla \cdot \vec{\mathbf{D}} = \rho_v$
(2) Ampere's law:	$\oint_C \vec{\mathbf{H}} \cdot d\vec{\ell} = \mathbf{I_c} + \mathbf{I_d} = \mathbf{I}$	$\nabla \times \vec{\mathbf{H}} = J + \frac{\partial D}{\partial t} = (\sigma + j\omega\varepsilon)\, \mathbf{E}$
(3) Faraday's law:	$\oint_{\mathbf{C}} \vec{\mathbf{E}} \cdot d\vec{\ell} = - \int_{\mathbf{S}} \frac{\partial \vec{\mathbf{B}}}{\partial \mathbf{t}} \cdot d\vec{s}$	$\nabla \times \vec{E} = -j\omega B$
(4) Gauss's law: (for magnetic fields)	$\oint_S \vec{\mathbf{B}} \cdot d\vec{s} = 0$	$\nabla \cdot \vec{\mathbf{B}} = 0$

The charge-current continuity relation or simply as the charge continuity equation (for conservation of electric charge) is

$$\nabla \cdot \vec{\mathbf{J}} = -\frac{\partial \rho_v}{\partial t}$$

In a static case none of the field quantities are function of time. Hence $\frac{\partial}{\partial t} = 0$. The Maxwell's equations reduce for the static case to the following form:

- $\nabla \cdot \vec{\mathbf{D}} = \rho_v$, $\nabla \times \vec{E} = 0$ (Electrostatics)
- $\nabla \cdot \vec{\mathbf{B}} = 0$, $\nabla \times \vec{\mathbf{H}} = \vec{\mathbf{J}}$ (Magnetostatics)

1.2 POISSON'S AND LAPLACE'S EQUATIONS

From the point form of Gauss's law, $\nabla . D = \rho_v$ (1)

The definition of D is $D = \varepsilon E$ (2)

The gradient relationship is $E = -\nabla V$ (3)

By the substitution of these two relations, we have

$$\nabla . D = \nabla . (\varepsilon E) = \nabla . [\varepsilon(-\nabla V)] = \rho_V$$

$$\nabla . \nabla V = -\frac{\rho_v}{\varepsilon} \quad \text{or} \quad \nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad (4)$$

For a homogeneous medium, ε is constant.
Equation (4) is *Poisson's equation*.

In Cartesian co-ordinates,

$$\nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z \qquad (5)$$

and

$$\nabla . A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial x} + \frac{\partial A_z}{\partial x} \qquad (6)$$

Therefore,

$$\nabla . \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 V = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = -\frac{\rho_v}{\varepsilon}$$ in Cartesian co-ordinates.

If $\rho_v = 0$ indicating zero volume charge density but allowing point charges, line charge and surface charge to exist at singular locations.

Then, $\nabla^2 V = 0$ (7)

This is *Laplace's equation*. The ∇^2 operation is called *Laplacian* of V.

The Laplace's equation in Cartesian co-ordinates is

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In other co-ordinates systems, the $\nabla^2 V$ is expressed by the relations below:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2}$$ Cylindrical co-ordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$ Spherical co-ordinates

Example 1:

Calculate the numerical values for V and ρ_v at a point P in free space if (a) $V=\frac{4yz}{x^2+1}$ at P(1,2,3) (b) $V=5\rho^2\cos 2\Phi$ at $P(\rho=3, \Phi=\pi/3, z=2)$.

Solution:

For each case substitute the coordinates and find V. Use appropriate formula for $\nabla^2 V$. Then, differentiate it and evaluate $\frac{-\rho_v}{\varepsilon}$ from $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$. Lastly, find ρ_v.

(a) $V=\frac{4 \times 2 \times 3}{1+1}$=12 v

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

Differentiating twice and substituting values of x, y, z, we yield $\nabla^2 V = -12$

$\rho_v=12\varepsilon_o=12\times 8.854\times 10^{-12}$=106.25 pico-colombs/$m^3$.

(b) V= $5 \times (3)^2 \times \cos(2\pi/3)$ = 22.5 volts

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial\phi^2}\right) + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\left(5\rho^2\cos2\phi\right)}{\partial\rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2(5\rho^2\cos2\phi)}{\partial\phi^2}\right) + \frac{\partial^2(5\rho^2\cos2\phi)}{\partial z^2}$$

$=20\cos2\phi - 20\cos2\phi+0 = 0.$

Therefore $\rho_v=0$

Example 2: (Rectangular Co-ordinates System)

Consider that the potential V is a function of x only.

The Laplace's equation reduces to $\dfrac{\partial^2 V}{\partial x^2} = 0$ (1)

Since V is not a function of y or z the partial derivative can be changed to ordinary derivative.

Integrating it, we get $\dfrac{dV}{dx} = A$ (2)

where A is a constant.

Integrating again, $V= Ax + B$ (3)

where B is another constant of integration.

These constants are to be evaluated using boundary conditions.

Let $V=V_1$ at $x=x_1$ and $V=V_2$ at $x=x_2$.

From (3) $V_1 = A x_1 + B$ and $V_2 = A x_2 + B$, solving them, we get

$A = \dfrac{V_1 - V_2}{x_1 - x_2}$ and $B = \dfrac{V_2 x_1 - V_1 x_2}{x_1 - x_2}$. Substituting for A and B in (3) we get

$V=\dfrac{V_1\left(x - x_2\right) - V_2\left(x - x_1\right)}{x_1 - x_2}$ (4)

If the boundary conditions are V=0 at x=0 and $V=V_o$ at x=d then $A=V_o/d$ and B=0, now

$V = \dfrac{V_o x}{d}$ (5)

In case of parallel plate capacitor of potential difference V_o and distance between plates **d** we can orient the plates such one is at x=0 and V=0 .

Using (5), we can find V.

From V find $\vec{E} = -\nabla V$. Find $\vec{D} = \varepsilon \vec{E}$. Find D at either capacitor plate.

D= $D_s = D_N a_N$, but $D_N = \rho_s$. Hence the charge Q can be found by surface integration over capacitor plate. $Q = \int_s \rho_s ds$. In the present case,

$$V = \frac{V_o x}{d}, \quad E = -\frac{V_o}{d}\hat{a}_x, \quad D = -\varepsilon \frac{V_o}{d}\hat{a}_x, \ Ds = D\Big|_{x=o} = -\varepsilon \frac{V_o}{d}\hat{a}_x, \ \hat{a}_N = \hat{a}_x$$

$D_N = -\varepsilon \frac{V_o}{d} = \rho_s$ and $Q = \int_s \frac{-\varepsilon V_o}{d} ds = -\varepsilon \frac{V_o S}{d}$ where S = Surface area.

$\therefore$ Capacitance $C = \frac{|Q|}{V_o} = \frac{\varepsilon S}{d}$.

Note: if the field varies with y or z co-ordinate, the working will be similar to the above example where x is replaced by y or z.

Example 3: (Cylindrical Co-ordinates System)
Consider variation of field with respect to ρ only.
The Laplace's equation is

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = 0 .$$

Converting the partial derivative to normal derivative, we yield

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0 .$$

Since ρ is in the denominator exclude the solution ρ=0.

Multiply by ρ, we get $\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$.

Integrating it, we yield $\left(\rho \frac{dV}{d\rho}\right) = A$ and thus $\left(\frac{dV}{d\rho}\right) = \frac{A}{\rho}$.

Integrating again, V= A ln ρ+B where A and B are constants of integration to be evaluated from boundary conditions.

Consider two co-axial cylinders of length L and radius a, b respectively with a potential difference V_o between them. Let b>a. V= V_o at ρ=a and V=0 at ρ=b.

Then the solution is $V = V_o \frac{\ln(b/\rho)}{\ln(b/a)}$.

From this we find E, where $\vec{E} = \frac{V_o}{\rho} \frac{1}{\ln(b/a)} \hat{a}_\rho$.

$D_{N(\rho=a)}=\frac{\varepsilon V_o}{a \ln(b/a)}$ and $Q=\frac{\varepsilon V_o 2\pi aL}{a \ln(b/a)}$.

Finally $C=\frac{2\pi\varepsilon L}{\ln(b/a)}$ is value of capacitance of coaxial capacitor.

Example 4: (Solution of Laplace's Equation by the Method of Separation of Variables or Product Solution)

This method is used when the potential function depends on two or more variables.
Let the potential be a function of x and y alone.

Then Laplace's equation is $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$.

We assume a solution of form V=XY where X is a function of x alone and Y is a function of y alone.

Then, $Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$

Because of independence of variables, we can use ordinary derivatives.

$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} = 0$. Dividing throughout by XY, we yield

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} = 0 \text{ or } \frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Y}\frac{d^2 Y}{dy^2}$$

Since X is a function of x alone and Y is a function of y alone the above equality is valid only if both the terms are equal to same constant.

$\frac{1}{X}\frac{d^2 X}{dx^2} = \alpha^2$ and $-\frac{1}{Y}\frac{d^2 Y}{dy^2} = \alpha^2$ where α^2 is called *separation constant*.

Now, we can rearrange the equations as $\frac{d^2 X}{dx^2} = \alpha^2 X$ and $\frac{d^2 Y}{dy^2} = -\alpha^2 Y$.

By solving these equations we get result. There are many types of solutions.
One solution is $X=c_1 e^{\alpha x} + c_2 e^{-\alpha x}$ and $Y=c_3 e^{j\alpha y} + c_4 e^{-j\alpha y}$ where c_1 to c_4 are constants to be evaluated from boundary conditions.

1.3 UNIQUENESS THEOREM

Within a closed region a solution to Laplace's equation or to Poisson's equation for the potential function V is **the only solution** that satisfies potential specified over the boundaries of the region.

Proof:

Assume we have two solutions for Laplace's equation, V_1 and V_2. Therefore,
$\nabla^2 V_1 = 0$, $\nabla^2 V_2 = 0$. $\therefore \nabla^2(V_1 - V_2) = 0$.

Each of the above solutions must satisfy the boundary conditions. Let V_b be the potential on boundary. Then $V_{1b,}$ value of V_1 on boundary and $V_{2b,}$ value of V_2 on boundary must be equal to $V_{b,}$ i.e. $V_{1b}= V_{2b}= V_b$ or $V_{1b}- V_{2b}=0$.

Consider the vector identity where V is scalar and D is vector.

$$\nabla.\left(V\vec{D}\right) \equiv V\left(\nabla.\vec{D}\right) + \vec{D}.(\nabla V)$$

Consider the scalar (V_1-V_2) and the vector $\nabla(V_1 - V_2)$. Then the equation will be

$$\nabla.[(V_1 - V_2)\nabla(V_1 - V_2)] \equiv (V_1 - V_2)[\nabla.\nabla(V_1 - V_2)] + \nabla(V_1 - V_2).\nabla(V_1 - V_2)$$

Integrate throughout the volume enclosed by the boundary surfaces specified, we yield

$$\int_{vol}\nabla.[(V_1 - V_2)\nabla(V_1 - V_2)]dv \equiv \int_{vol}(V_1 - V_2)[\nabla.\nabla(V_1 - V_2)]dv + \int_{vol}[\nabla(V_1 - V_2)]^2 dv\,.$$

Using Divergence theorem, the equation is rewritten as

$$\int_{vol}\nabla.[(V_1 - V_2)\nabla(V_1 - V_2)]dv \equiv \oint_s[(V_{1b} - V_{2b})\nabla(V_{1b} - V_{2b})].ds = 0$$

One of the factors $\nabla.\nabla(V_1 - V_2) = \nabla^2(V_1 - V_2) = 0$ by hypothesis and that integral is zero. So the remaining integral also must be zero, i.e. $\int_{vol}[\nabla(V_1 - V_2)]^2 dv = 0$.

For the equality to be true, $[\nabla(V_1 - V_2)]^2=0$.

So, $\nabla(V_1 - V_2) = 0$.

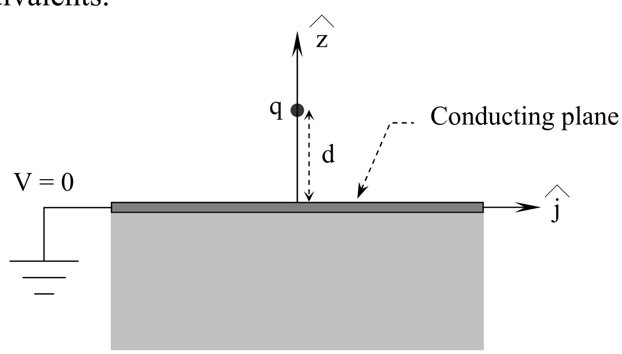
Since the gradient is zero everywhere, (V_1-V_2) cannot change with co-ordinates and must be constant. Considering the points on boundary, V_1-V_2=V_{1b}-V_{2b}=0.

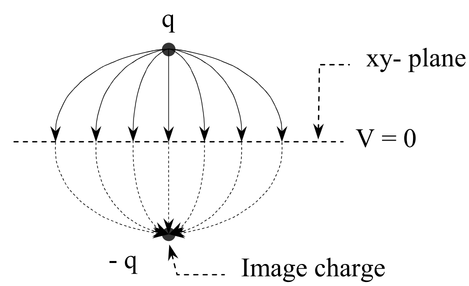
Therefore V_1=V_2. Hence the two solutions are identical.

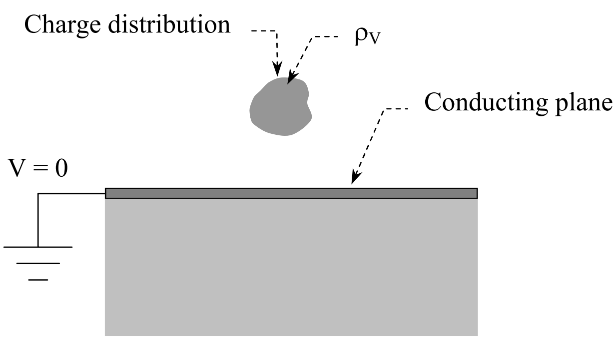
1.4 METHOD OF IMAGES

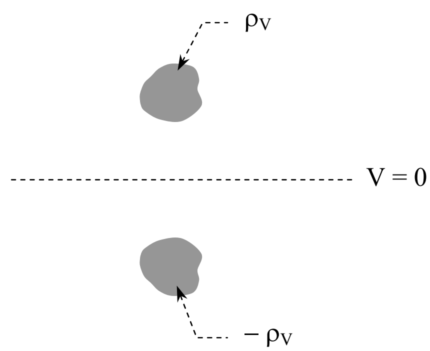
There are problems in electrostatics that are difficult to solve using methods like Coulombs law, Gauss's law, and directly solving Poisson's or Laplace's equation. One such problem is finding the potential distribution of a point charge situated at a fixed distance above an infinite perfectly conducting ground plane. Such problems can be solved using a technique known as the *Method of Images*.

The method of images is based on the theory that "*any given charge distribution above an infinite, perfectly conducting plane is electrically equivalent to the combination of the given charge configuration and its image configuration, with the conducting plane removed*". The diagram below shows some examples of replacing charge distributions with their image equivalents.



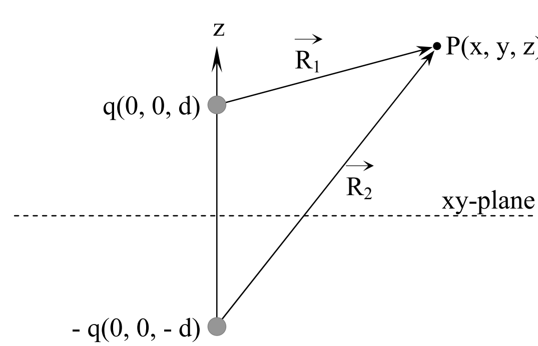






Example 1:
Determine the surface charge density distribution induced on a perfectly conducting grounded plane by a point charge q situated at a perpendicular distance d from the plane.

Solution:
To find the surface charge distribution we have to find the normal component of the electric field intensity at any point on the plane. First find the field intensity E at any point P(x, y, z) above the plane.



$$\vec{E} = \frac{1}{4\pi\,\varepsilon_0}\left(q\,\frac{\vec{R}_1}{R_1^3} - q\,\frac{\vec{R}_2}{R_2^3}\right)$$

$$= \frac{1}{4\pi\varepsilon_0} q \left[\frac{x\hat{a}_x + y\hat{a}_y + (z-d)\hat{a}_z}{[x^2+y^2+(z-d)^2]^{\frac{3}{2}}} - \frac{x\hat{a}_x + y\hat{a}_y + (z+d)\hat{a}_z}{[x^2+y^2+(z+d)^2]^{\frac{3}{2}}} \right] \quad \text{for } z \geq 0.$$

On the conducting plane z=0, the electric field intensity at any point P(x, y, 0) on the conducting plane is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} q \left[\frac{-d\hat{a}_z}{[x^2+y^2+d^2]^{\frac{3}{2}}} - \frac{d\hat{a}_z}{[x^2+y^2+d^2]^{\frac{3}{2}}} \right] = -\frac{1}{2\pi\varepsilon_0} q \frac{d\hat{a}_z}{[x^2+y^2+d^2]^{\frac{3}{2}}}.$$

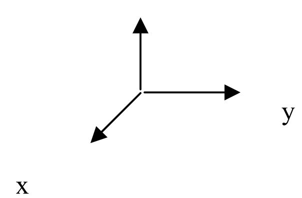
Since the unit vector normal to the plane is $\hat{k}$, we yield

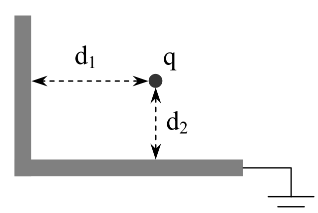
$$E_n = \frac{1}{\varepsilon_0}\rho_s = -\frac{1}{2\pi\varepsilon_0} q \frac{d\hat{a}_z}{[x^2+y^2+d^2]^{\frac{3}{2}}} \bullet \hat{a}_z = -\frac{1}{2\pi\varepsilon_0} q \frac{d}{[x^2+y^2+d^2]^{\frac{3}{2}}}$$

or $\rho_s = -\frac{1}{2\pi} q \frac{d}{[x^2+y^2+d^2]^{\frac{3}{2}}}$.

Example 2:

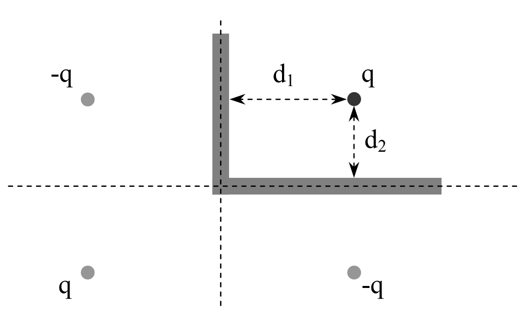
A positive point charge q is located at distances d_1 and d_2, respectively, from two perfectly conducting perpendicular half-planes (see diagram). Find the force on q due to the charges induced on the planes.

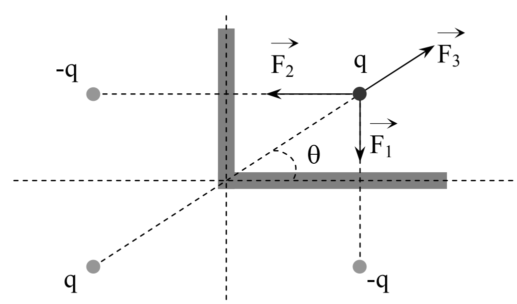




Solution:

The equivalent image charge arrangement is as shown in the diagram below:





The three image charges are required to ensure that the potential at both the horizontal and the vertical half-planes is zero. These three image charges exert forces $\vec{F_1}$, $\vec{F_2}$, and $\vec{F_3}$ on the charge q, and the resultant force acting on q is $\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3}$. Using Coulomb's law, we find:

$$\vec{F_1} = -\,\hat{a}_z\, q^2 \frac{1}{4\pi\,\varepsilon_0} \frac{1}{(2\,d_2)^2}$$

$$\vec{F_2} = -\,\hat{a}_y\, q^2 \frac{1}{4\pi\,\varepsilon_0} \frac{1}{(2\,d_1)^2}$$

$$\vec{F_3} = q^2 \frac{1}{4\pi\,\varepsilon_0} \frac{1}{(2\,d_1)^2 + (2\,d_2)^2} (\hat{a}_y \cos\theta + \hat{a}_z \sin\theta)$$

$$= q^2 \frac{1}{4\pi\,\varepsilon_0} \frac{1}{[(2\,d_1)^2 + (2\,d_2)^2]^{\frac{3}{2}}} (\hat{a}_y\, 2\,d_1 + \hat{a}_z\, 2\,d_2).$$

Therefore, the resultant force is

$$\vec{F} = q^2 \frac{1}{16\pi\,\varepsilon_0} \{\hat{a}_y \left[\frac{d_1}{(d_1^2 + d_2^2)^{\frac{3}{2}}} - \frac{1}{d_1^2}\right] + \hat{a}_z \left[\frac{d_2}{(d_1^2 + d_2^2)^{\frac{3}{2}}} - \frac{1}{d_2^2}\right]\}.$$