Chapter 4 Time-Domain Analysis of Control Systems

4.1 INTRODUCTION

The ability to adjust the transient and steady-state response of a control system is a beneficial outcome of the design of a feedback system. Since time is used as an independent variable in most of control systems, it is usually of interest to evaluate the state and output responses with respect to time, or simply the time response.

In the analysis problem we will use selected input signals to test the response of control system. This response will be characterized by a selected set of response measures. In this chapter, we will strive to delineate a set of quantitative performance measures that adequately represent the performance of the control systems.

4.2 TIME RESPONSE AND TEST SIGNALS

The time response of a control system is usually divided into two parts: the transient response and the steady-state response. Let $y(t)$ denote the time response of a continues-data system; then, in general, it can be written as

$$y(t) = y_t(t) + y_{ss}(t) \qquad (4.1)$$

where $y_t(t)$ denotes the transient response and $y_{ss}(t)$ denotes the steady-state response.

In control systems, transient response is defined as the part of the time response that goes to zero as time becomes very large. Thus $y_t(t)$ has the property

$$\lim_{t\to\infty} y_t(t) = 0 \qquad (4.2)$$

The steady-state response is simply the part of the total response that remains after transient has died out. All real stable systems exhibit transient phenomena to some extent before the steady state is reached.

In the design problem, specifications are usually given in terms of the transient and steady-state performance, and controllers are designed so that the specifications are all met by the design system.

Since it is difficult to design a control system so that it will perform satisfactory for all possible forms of input signals, it is necessary, for purpose of analysis and design, to assume some basic types of test signals properly for the prediction of system's performance to other more complex inputs.

1. Step-Function Input

The step-function input represents an instantaneous change in the reference input. The mathematical representation of a step function of magnitude R is

$$r(t) = \begin{cases} R & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Mathematically, $r(t) = Ru_s(t)$, where $u_s(t)$ is the unit-step function. The step function is shown in Figure 4.1 (a).

2. Ramp-Function Input

The ramp function is a signal that changes constantly with time. Mathematically, a ramp function is represented by

$$r(t) = Rtu_s(t)$$

where R is a real constant. The ramp function is shown in Figure 4.1 (b).

3. Parabolic-Function Input

The parabolic function represents a signal that is one order faster than the ramp function. Mathematically, it is represented as

$$r(t) = \frac{Rt^2}{2} u_s(t)$$

The Parabolic function is shown in Figure 4.1 (c).

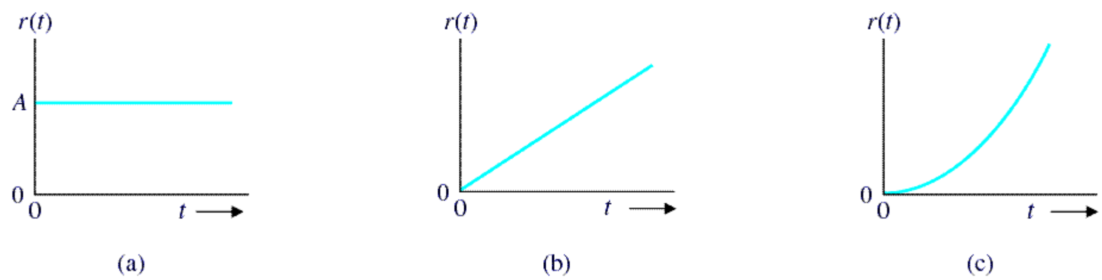

Figure 4.1 Time-domain test input signals: (a) Step, (b) Ramp, (c) Parabolic

4.3 UNIT-STEP RESPONSE AND TIME-DOMAIN SPECIFICATIONS

For linear control systems, the time response is characterized by using the unit step-input. The response of the control system to the unit step-input is called unit-step response. Figure 4.2 illustrate a typical unit-step response of a linear control system.

With reference to unit-step response, the following performance criteria (parameters) are defined:

1. Maximum overshoot

Let y_{max} denotes the maximum value of $y(t)$ and y_{ss} be the steady-state value of $y(t)$ and $y_{max} \geq y_{ss}$. The maximum overshoot of $y(t)$ is defined as,

Maximum overshoot = $y_{max} - y_{ss}$

$$\text{Percent maximum overshoot} = \frac{\text{maximum overshoot}}{y_{ss}} \times 100\% \qquad (4.3)$$

2. Delay time

The delay time, t_d is defined as the time required for the step response to reach 50% of its final value.

3. Rise time

The rise time, t_r is defined as the time required for the step response to rise from 10 to 90 percent of its final value.

4. Settling time

The settling time, t_s is defined as the time required for the step response to reach and stay within a specified percentage (5%) of its final value.

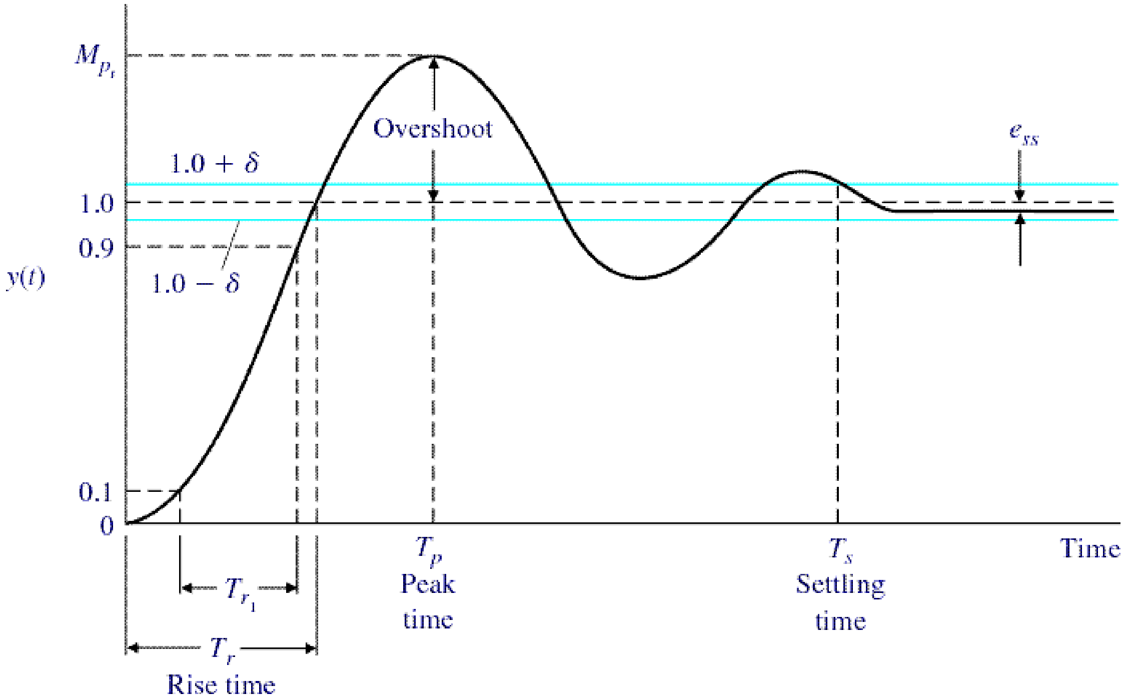

Figure 4.2 Step response of a control system

Analytically, these quantities are difficult to establish, except for simple systems lower than the third order.

4.4 TRANSIENT RESPONSE OF A PROTOTYPE SECOND-ORDER SYSTEM

Although true second-order control systems are rare in practice, their analysis generally helps to form a basis for the understanding of analysis and design of higher-order systems, especially the ones that can be approximated by second-order systems.

Consider that a second-order control system with unity feedback is represented by the block diagram shown in Figure 4.3. The open-loop transfer function of the system is

$$G(s) = \frac{\omega_n^2}{s(s + 2\xi\omega_n)} \tag{4.5}$$

where ξ and ω_n are real constants. The closed-loop transfer function of the system is

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{4.6}$$

The characteristic equation of the prototype second-order system is obtained by setting the denominator of Eq. 4.6 to zero

$$\Delta(s) = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad (4.7)$$

As we shall see later, the system is *stable* (*bounded output for bounded input*) if the roots of the characteristic equation locate on the left half of s-plane, and *marginally stable* (*oscillation for a bounded input*) if the characteristic equation has simple roots on the imaginary axis with all other roots in the left half of s-plane. For an *unstable* (*unbounded output for any bounded input*) system the characteristic equation has at least one root in the right half of the s-plane or it has a repeated $j\omega$ roots.

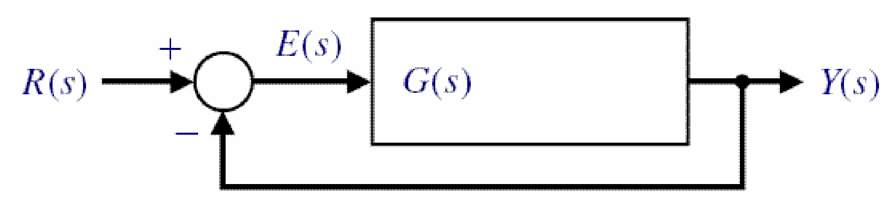

Figure 4.3 Prototype Second-order control system

For a unit-step input, R(s) = 1/s, the output response is given as

$$Y(s) = \frac{\omega_n^{\ 2}}{s(s^2 + 2\zeta\omega_n s + \omega_n^{\ 2})} \qquad (4.8)$$

By taking inverse Laplace transform, we obtain the unit step response of the control system

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_n \sqrt{1-\zeta^2}\, t + \cos^{-1}\zeta\right) \qquad t \ge 0 \qquad (4.9)$$

Figure 4.4 shows the unit-step response of the second-order system for various values of ζ. It may be noted that the response becomes more oscillatory with larger overshoot as ζ decreases.

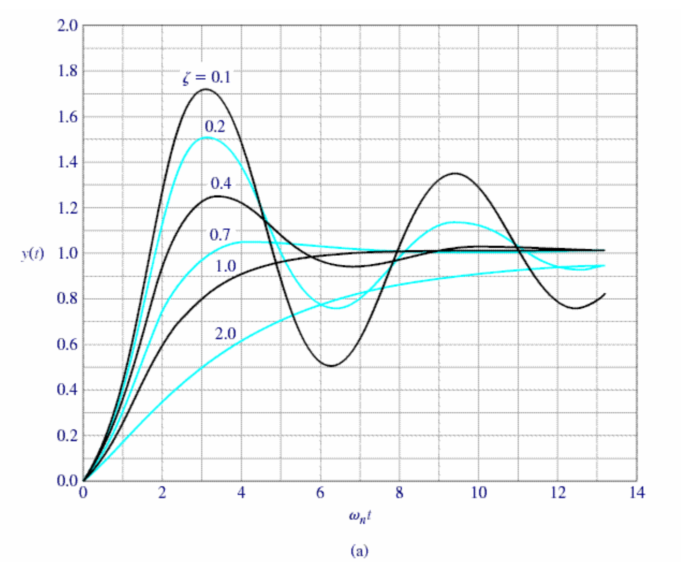

Figure 4.4 Unit-step response of second-order system with various ξ values

4.4.1 Damping Ratio and Damping Factor

The effects of the system parameters ξ and ω_n on the step response $y(t)$ can be studied by referring to the roots of the characteristic equation in Eq. (4.7). The roots can be expresses as

$$\begin{aligned} s_1, s_2 &= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \\ &= -\alpha + j\omega \end{aligned} \tag{4.10}$$

where $$\alpha = \xi\omega_n \tag{4.11}$$

and $$\omega = \omega_n\sqrt{1-\zeta^2} \tag{4.12}$$

The physical significance of ξ and α is now investigated. As seen from Eq. (4.9) the factor $\alpha = \zeta\omega_n$ appears as a constant multiplied by t in the exponential term of the response $y(t)$. Therefore, α controls the rate of rise or decay of the unit-step response $y(t)$. In other words, α controls the "damping" of the system and is called *damping factor*. The inverse of α , $1/\alpha$ is proportional to the time constant of the system. When $\zeta = 1$, the oscillations disappear and the system is said to be critically damped. Under this condition $\alpha = \omega_n$. Thus, we can regard ζ as

$$\zeta = \frac{\alpha}{\omega_n} = \frac{actual\ damping\ factor}{damping\ factor\ at\ the\ critical\ damping} \tag{4.13}$$

When $\zeta < 1$, the system is under-damped and when $\zeta > 1$, the system is over-damped.

4.4.2 Natural Undamped Frequency

The parameter ω_n is defined as the natural undamped frequency. As seen from equation (4.10), when $\zeta = 0$, the roots of the characteristic equation are imaginary. Thus, the unit-step response of the system becomes purely oscillatory with angular frequency of ω_n. For $0 < \zeta < 1$, the imaginary parts of the roots have the magnitude of the actual (damped) frequency of oscillation.

Thus, $$\omega = \omega_n\sqrt{1-\zeta^2}$$

Figure 4.5 illustrates the relationships between the location of the characteristic equation roots and α, ξ, and ω_n.

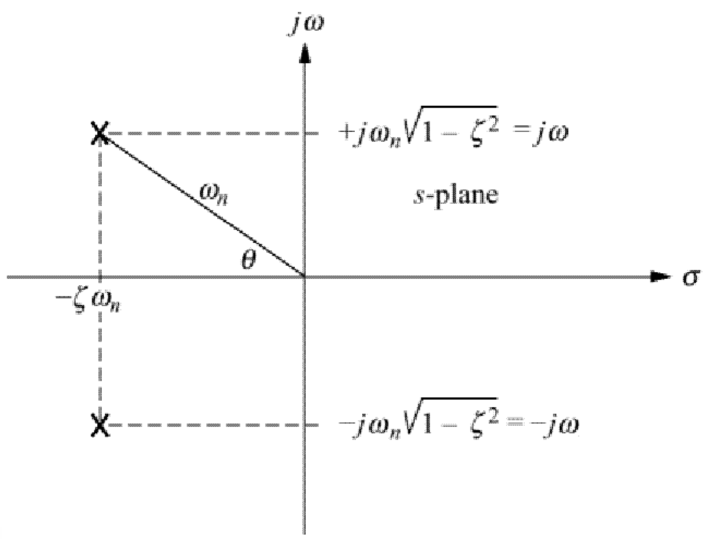

Figure 4.5 The relationship between the characteristic equation roots and α, ξ, and ω

The effect of the characteristic equation roots on the damping of the second-order system is illustrated in Figure 4.6

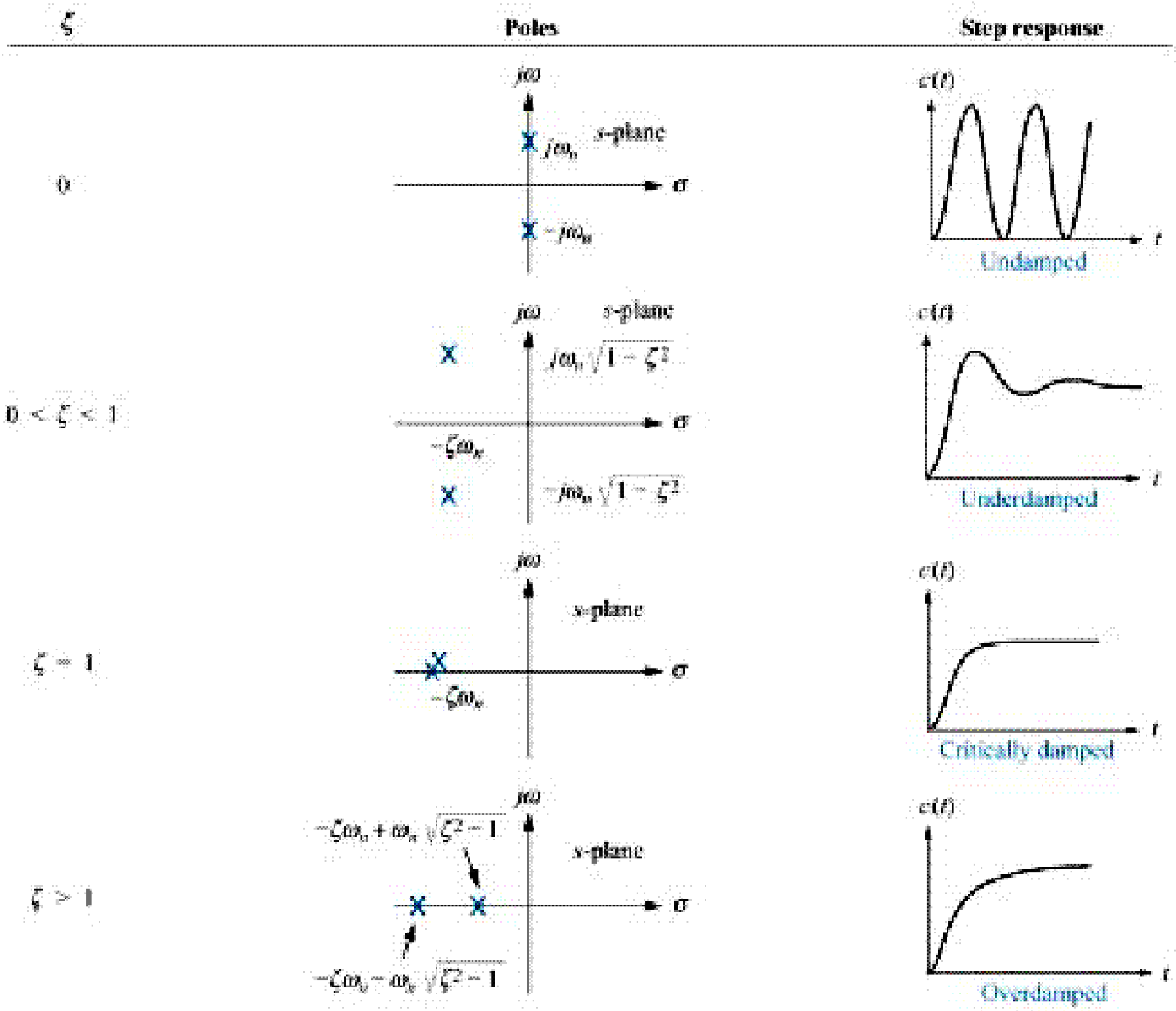

Figure 4.6 Step-response comparisons for various characteristic equation – root locations in the s-plane

4.4.3 Analytical Expression for Maximum Overshoot

By taking the derivative of Eq. (4.9) with respect to time *t* and setting the result to zero, we get

$$\frac{dy(t)}{dt} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\omega_n . \sqrt{1-\zeta^2} . t \qquad (4.14)$$

$$\omega_n \sqrt{1-\zeta^2}\, t = n\pi \qquad n = 1,2,3,\ldots$$

From which we get

$$t = \frac{n\pi}{\omega_n \sqrt{1-\zeta^2}} \qquad n = 1,2,3,\ldots \qquad (4.15)$$

For the unit-step responses shown in Fig. 4.4, the first overshoot is the maximum overshoot. This corresponds to n = 1 in Eq. (4.15). Thus, the time at which the maximum overshoot occurs is

$$t_{\max} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} \tag{4.16}$$

With reference to Fig. 4.4, the overshoots occur at odd values of n, that is, n =1, 3, 5, …, and undershoots occur at even values of n.

The magnitude of the overshoot and undershoots can be determined by subistituting Eq. (4.14) into Eq. (4.9). This results in $y(t)_{max}$ or $y(t)_{min}$. Therefore

$$\text{maximum overshoot} = y_{max} - 1 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{4.17}$$

and the percent maximum overshoot is

$$\text{percent maximum overshoot} = 100e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{4.18}$$

The relationship between the percent maximum overshoot and the damping ratio, given in Eq. (4.18) is plotted in Figure 4.7.

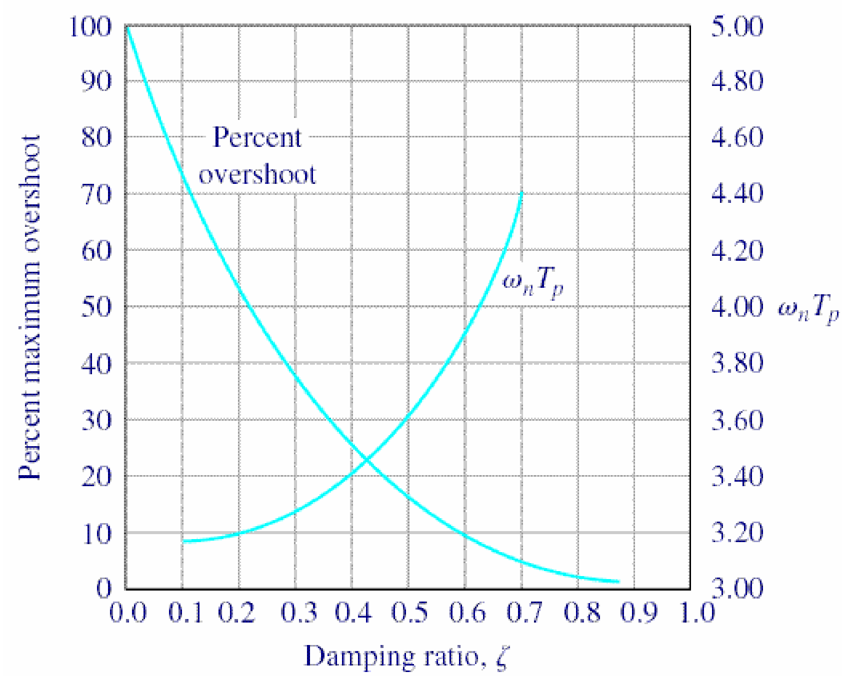

Figure 4.7 The relationship between the percent maximum overshoot and the damping ratio

4.4.4 Delay Time and Rise Time

It is more difficult to determine the exact analytical expressions of the delay time t_d and rise time t_r, and settling time t_s. However, we can utilize the linear approximation

$$t_d \cong \frac{1+0.7\zeta}{\omega_n} \qquad 0 < \zeta < 1.0 \tag{4.19}$$

The plot of $\omega_n t_r$ versus ξ is shown in Figure 4.8. This relation can be approximated by a straight line over a limited range of ξ:

$$t_r = \frac{0.60 + 2.16\zeta}{\omega_n} \qquad 0 < \zeta < 1 \tag{4.20}$$

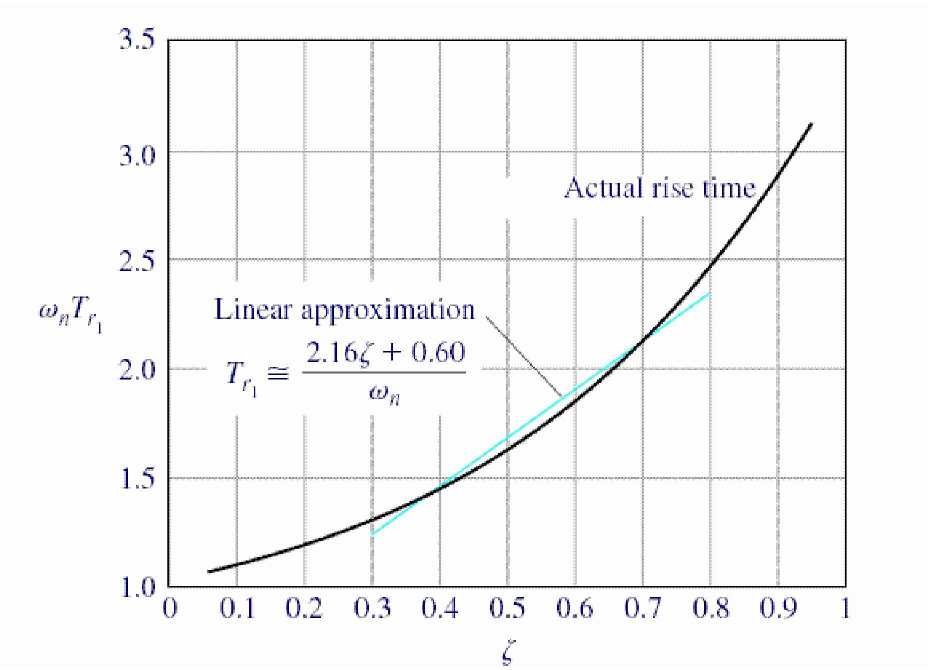

Figure 4.8 Normalized rise time versus ξ for the prototype second-order system

From this discussion, the following conclusions can be made:

1. t_r and t_d are proportional to ξ and inversely proportional to ω_n.
2. Increasing (decreasing) the natural undamped frequency ω_n will reduce (increase) t_r and t_d.

In regard to the settling time t_s, it can be approximated as

$$t_s \cong \frac{3.2}{\zeta\omega_n} \qquad 0 < \zeta < 0.69 \tag{4.21}$$

and

$$t_s = \frac{4.5\zeta}{\omega_n} \qquad \zeta > 0.69 \tag{4.22}$$

We can summarize the relationships between t_s and the system parameters as follows:

1. For $\xi < 0.69$, the settling time is inversely proportional to ξ and ω_n. A practical way of reducing the settling time is to increase ω_n while holding ξ constant.
2. For $\xi > 0.69$, the settling time is proportional to ξ and inversely proportional to ω_n. Again, t_s can be reduced by increasing ω_n.

4.5 STABILITY OF LINEAR CONTROL SYSTEMS

The transient response of a feedback control system is of primary interest and must be investigated. A stable system is defined as a system which gives a bounded output in response to a bounded input.

The concept of stability can be illustrated by considering a circular corn placed on a horizontal surface, Fig. 4.9 and Fig. 4.10.

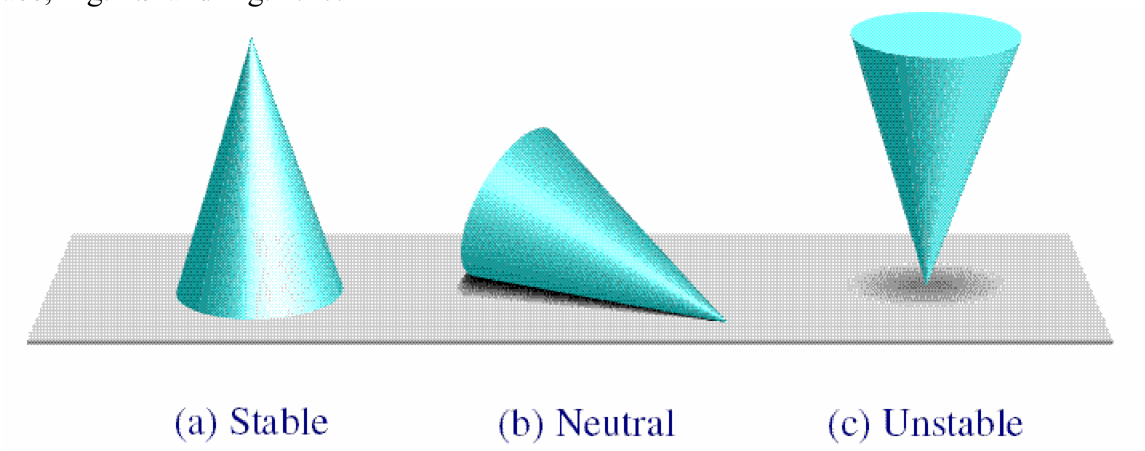

Figure 4.9 The stability of a cone

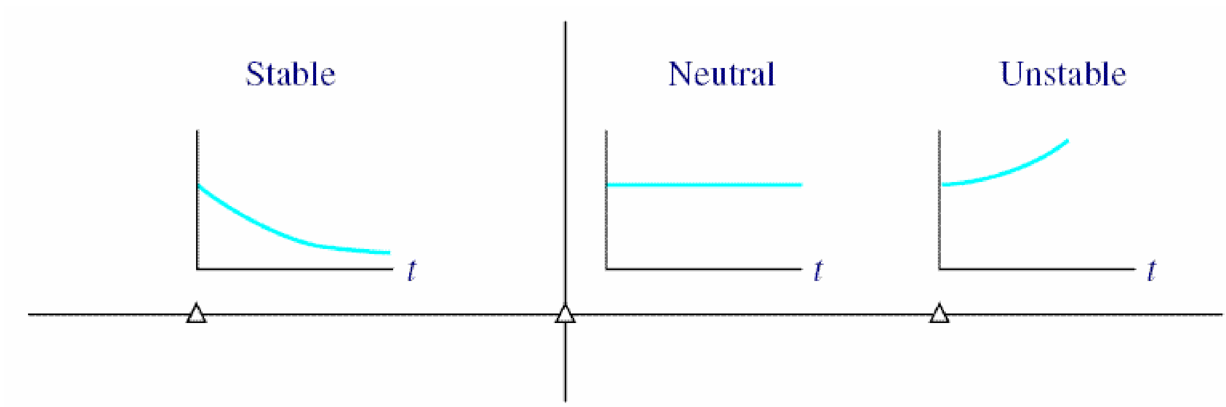

Figure 4.10 Stability in the s-plane

The stability of a dynamic system is defined in a similar manner. Let $u(t)$, $y(t)$, and $g(t)$ be the input, output, and impulse response of a linear time-invariant system, respectively. The output of the system is given by the convolution between the input and the system's impulse response. Then

$$y(t) = \int_0^{\infty} u(t-\tau)g(\tau)d\tau \tag{4.23}$$

This response is bounded (*stable system*) if and only if the absolute value of the impulse response, $g(t)$, integrated over an infinite range, is finite. That is

$$\int_0^{\infty} |g(\tau)| d\tau \quad < \infty \tag{4.24}$$

Mathematically, Eq. (4.24) is satisfied when the roots of the characteristic equation, or the poles of $G(s)$, are all located in the left-half s-plane.

A system is said to be *unstable* if any of the characteristic equation roots locates in the right-half s-plane. When the characteristic equation has simple roots on the jω-axis and none in the right-half plane, we refer to the system as *marginally stable*.

The following table illustrates the stability conditions of linear continuous system with reference to the locations of the roots of the characteristic equation.

Stability Condition	Roots Values
Stable	All the roots are in the left-half s-plane
Marginally stable or marginally unstable	At least one simple root, and no multiple roots on the jω-axis; and no roots in the right-half s-plane.
Unstable	At least one simple root in the right-half s-plane or at least one multiple-order root on the jω-axis.

Table 4.1 Stability Conditions of LTI System

The following examples illustrate the stability conditions of systems with reference to the poles of the closed-loop transfer function M(s).

$M(s)=\dfrac{20}{(s+1)(s+2)(s+3)}$	Stable
$M(s)=\dfrac{20(s+1)}{(s-1)(s^2+2s+2)}$	Unstable due to the pole at $s = 1$
$M(s)=\dfrac{20(s-1)}{(s+2)(s^2+4)}$	Marginally stable or marginally unstable due to $s = \pm j2$.
$M(s)=\dfrac{10}{(s^2+4)^2(s+10)}$	Unstable due to the multiple-order pole at $s = \pm j2$.

4.5.1 Methods of Determining Stability

The discussion in the proceeding sections lead to the conclusion that the stability of linear time-invariant system can be determined by checking on the location of the roots of the characteristic equation. When the system parameters are all known, the roots of the characteristic equation can be solved by means of a root-finding computer program. For example the M-file roots(a) of MATLAB.

For design purposes, there will be unknown or variable parameter embedded in the characteristic equation, and it will be feasible to use the root-finding programs. The method outlined below is well known for the determination of stability of LTI system without involving root solving.

4.5.1.1 Routh-Hurwitz Criterion

The Routh-Hurwitz criterion represents a method of determining the location of zeros of a polynomial with constant real coefficients with respect to the left and right half of the s-plane, without actually solving for the zeros.

Consider that the characteristic equation of a linear time-invariant SISO system is of the form

$$F(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0 \qquad (4.25)$$

where all the coefficients are real. In order that Eq. (4.25) not has roots in the right half of *s*-plane, it is *necessary* and *insufficient* that the following conditions hold:

1. All the coefficients of the equation have the same sign
2. None of the coefficients vanishes

However, these conditions are not sufficient, for it is quite possible that an equation with all its coefficients nonzero and with the same sign still may not have all the roots in the left half of the *s*-plane.

The first step in the Routh-Hurwitz criterion is to arrange the coefficients of the Eq. (4.25) as follows:

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \end{array}$$

Further rows of the schedule are then completed as follows:

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ s^0 & h_{n-1} & & & \end{array}$$

where

$$b_{n-1} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_{n-3} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_{n-1} = -\frac{1}{b_{n-1}}\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}$$

and so on. Once The Routh's tabulation has been completed, we investigate the signs of the coefficients in the first column of the tabulation.

The roots of the equation are all in the left half of the *s*-plane if all the elements of the first column of the Routh's tabulation are of the same sign. The number of changes of signs in the elements of the first column equals the number of roots with positive real parts or in the right-half *s*-plane.

Example 4.1

Consider the equation

$$(s-2)(s+1)(s-3) = s^3 - 4s^2 + s + 6 = 0$$

This equation has one negative coefficient. Thus, we know without applying Routh's test that not all the roots of the equation are in the left-half s-plane. In fact, from the factored form of the equation, we know that there are two roots in the right-half s-plane, at s = 2 and s = 3. For purpose of illustrating the Routh's Tabulation, it is made as follows:

s^3	1	1
s^2	-4	6
s^1	2.5	0
s^0	6	0

Since there are two sign changes in the first column of the tabulation, the equation has two roots located in the right-half s-plane.

Example 4.2

Consider the equation

$$2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

Since this equation has no missing terms and the coefficients are all of the same sign, it satisfies the necessary conditions for not having roots in the right half or on the imaginary axis of the s-plane. However, since these conditions are necessary but not sufficient, we have to check Routh's tabulation.

s^4	2	3	10
s^3	1	5	0
s^2	-7	10	0
s^1	6.43	0	0
s^0	10	0	0

Since there are two changes in the first column of the tabulation, the equation has two roots in the right half of the s-plane.

4.5.1.2 Special Cases When Routh's Tabulation Terminates Prematurely

Depending on the coefficients of the equation, the following difficulties may occur that prevent Routh's tabulation from completing properly:

1. The first element in any one row of Routh's tabulation is zero, but the others are not.
2. The elements in one row of Routh's tabulation are all zero.

In the first case we replace the zero element in the first column by an arbitrary small positive number ε, and then proceed with Routh's tabulation.
This is illustrated by the following example.

Example 4.3

Consider the characteristic equation of a linear system:

$$s^4 + s^3 + 2s^2 + 2s + 3 = 0$$

Since all the coefficients are nonzero and of the same sign, we need to apply the Routh-Hurwitz criterion. Routh's tabulation is carried out as follows:

s^4	1	2	3
s^3	1	2	0
s^2	0	3	

Since the first element of the s^2 row is zero, the element in the s^1 row would all be infinite. To overcome this difficulty, we replace the zero in the s^2 row by a small positive number ε and then proceed with the tabulation.

s^2	ε	3
s^1	$\cong -\dfrac{3}{\varepsilon}$	0
s^0	3	0

Since there are two sign changes in the first column of Routh's tabulation, the equation has two roots in the right-half s-plane.

In the second special case, when all the elements in one row of Routh's tabulation are zeros before the tabulation is properly terminated, it indicates that one or more of the following conditions may exist:

1. The equation has at least one pair of real roots with equal magnitude but opposite signs.
2. The equation has one or more pairs of imaginary roots.
3. The equation has pairs of complex-conjugate roots forming symmetry about the origin of the s-plane (e.g. $s = -1 \pm j1$, $s = 1 \pm j1$).

The situation with the entire row of zeros can be remedied by using the auxiliary equation $A(s) = 0$, which is formed from the coefficients of the row just above the row of zeros in Routh's tabulation. The roots of the auxiliary equation also satisfy the original equation.

To continue with Routh's tabulation when a row of zeros appears, we conduct the following steps:

1. For the auxiliary equation $A(s) = 0$ by use of the coefficients from the row just preceding the row of zeros.
2. Take the derivative of the auxiliary equation with respect to s; this gives $dA(s)/ds = 0$.
3. Replace the row of zeros with the coefficients of $dA(s)/ds = 0$.
4. Continue with Routh's tabulation in the usual manner.

Example 4.4

Consider the following characteristic equation of a linear control system:

$$s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$$

Routh's tabulation is

$$\begin{array}{llll} s^5 & 1 & 8 & 7 \\ s^4 & 4 & 8 & 4 \\ s^3 & 6 & 6 & 0 \\ s^2 & 4 & 4 & \\ s^1 & 0 & 0 & \end{array}$$

A(s) = $4s^2 + 4 = 0$
The derivative of A(s) with respect to s is
$d\text{A}(s)/ds = 8s = 0$

From which the remaining portion of the Routh's tabulation is

$$\begin{array}{lll} s^1 & 8 & 0 \\ s^0 & 4 & \end{array}$$

Since there are no sign changes in the first column, the system is stable. Solving the auxiliary equation A(s) = 0, we get the two roots at s = j and s = -j, which are also two of the roots of the characteristic equation. Thus the equation has two roots on the jω-axis, and the system is marginally stable. These imaginary roots caused the tabulation to have an entire row of zeros in the s^1 row.

Example 4.5
Consider that a third-order control system has the characteristic equation

$$s^3 + 3408.3s^2 + 1204\times10^3 s + 1.5\times10^7 k = 0$$

Determine the crucial value of k for stability.

Routh's tabulation is

$$\begin{array}{lll} s^3 & 1 & 1204\times10^3 \\ s^2 & 3408 & 1.5\times10^7 k \\ s^1 & -\dfrac{1.5\times10^7 k - 3408\times1204\times10^3}{3408} & 0 \\ s^0 & 1.5\times10^7 & \end{array}$$

For the system to be stable, all the coefficients in the first column must have the same sign. This lead to the following conditions:

$$-\frac{1.5\times10^7 k - 410.36\times10^7}{3408} > 0$$

Therefore, the condition of k for the system to be stable is

$$0 < k < 273.57$$

If we let k = 273.57, the characteristic equation will have two roots on the jω-axis.
To find these roots, we substitute k = 273.57 in the auxiliary equation, as follows:

$A(s) = 3408.3s^2 + 4.1036 \times 10^9 = 0$

which has roots at s = j1097.27 and s = -j1097.27. Thus if the system operate with k = 273.57, the system response will be an undamped sinusoid with a frequency of 1097.27 rad/sec.

4.6 STEADY STATE ERROR

One of the objectives of most control systems is that the system output response follows a specific reference signal accurately in the steady state. Steady-state error is the difference between the output and the reference in the steady state. Steady-state errors in control systems are almost unavoidable and generally derive from the imperfections, frictions, and the natural composition of the system. In the design problem, one of the objectives is to keep the steady-state error below a certain tolerable value.

4.6.1 Definition of the Steady-State Error with Respect to System Configuration

Let us refer to the closed-loop system shown in Figure 4.11, where *r(t)* is the input, *e(t)* is the actuating signal, and *y(t)* is the output.

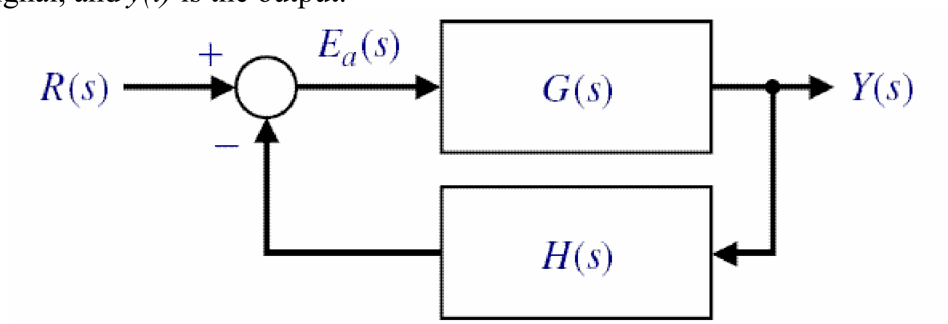

Figure 4.11 Closed-Loop Control System

The error of the system may be defined as:

$$e(t) = \text{reference signal - y(t)} \qquad (4.26)$$

where the reference signal is the signal that the output is to track. When the system has unity feedback [i.e. H(s) = 1], the error is simply

$$e(t) = r(t) - y(t)$$

The steady-state error is defined as

$$\begin{aligned} e_{ss} &= \lim_{s\to\infty} e(t) = \lim_{s\to\infty} sE(s) \\ &= \lim_{s\to\infty} \frac{sR(s)}{1+G(s)} \end{aligned} \qquad (4.27)$$

Clearly, e_{ss} depends on the characteristics of $G(s)$. More specifically, e_{ss} depends on the number of poles that $G(s)$ has at s = 0. This number is known as the system type. Figure 4.12 shows steady state errors for different input functions.

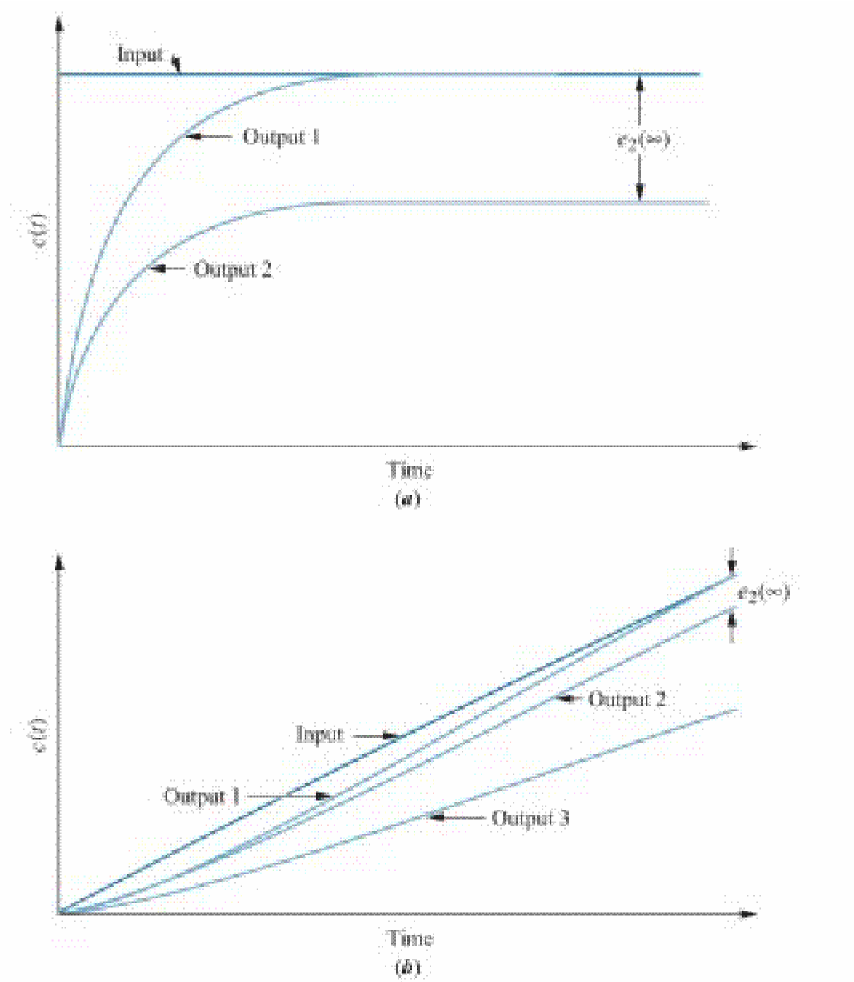

Figure 4.12 Steady-state errors: (a) step input, (b) ramp input

Now, let us investigate the effects of the types of inputs on the steady-state error.

4.6.2 Steady-State Error of System with a Step-Function Input

When the input $r(t)$ to a control system with unity-feedback is a step function with magnitude R, then $R(s) = R/s$ and the steady-state error is written from Eq. (4.27) as

$$e_{ss} = \lim_{s\to\infty} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{R}{1+G(s)} = \frac{R}{1+\lim_{s\to 0} G(s)} \tag{4.28}$$

For convenience, we define

$$K_p = \lim_{s\to 0} G(s)$$

as the *step-error constant*. Then Eq. (4.28) becomes

$$e_{ss} = \frac{R}{1+K_p} \tag{4.29}$$

We can summarize the steady-state error due to a step-function input as follows:

- Type 0 system: $e_{ss} = \dfrac{R}{1+K_p}$ = constant
- Type 1 or higher system: $e_{ss} = 0$

4.6.3 Steady-State Error of System with a Ramp-Function Input

When the input to the unity-feedback control system is a ramp function with amplitude *R*,

$$r(t) = Rtu_s(t)$$

where *R* is a real constant, the Laplace transform of *r*(*t*) is $R(s) = \dfrac{R}{s^2}$

The steady-state error is written using Eq. (4.27) as follows:

$$e_{ss} = \lim_{s\to 0} \frac{R}{s + sG(s)} = \frac{R}{\lim_{s\to 0} sG(s)} \tag{4.30}$$

We define the *ramp-error constant* as $k_v = \lim_{s\to 0} sG(s)$

Then Eq. (4.30) becomes $e_{ss} = \dfrac{R}{k_v}$ (4.31)

The following conclusions may be stated with regard to the steady-state error of a system with ramp input:

- Type 0 system: $e_{ss} = \infty$
- Type 1 system: $e_{ss} = R/k_v$ = *constant*
- Type 2 or higher system: $e_{ss} = 0$

4.6.4 Steady-State Error of System with a Parabolic Input

When the input is described by the standard parabolic form

$$r(t) = \frac{Rt^2}{2} u_s(t)$$

The Laplace transform of *r*(*t*) is $R(s) = \dfrac{R}{s^3}$

The steady-state error of the system is

$$e_{ss} = \frac{R}{\lim_{s\to 0} s^2 G(s)} \tag{4.32}$$

Defining the parabolic-error constant as

$$k_a = \lim_{s\to 0} s^2 G(s) \tag{4.33}$$

the steady-state error becomes $e_{ss} = \dfrac{R}{k_a}$ (4.34)

The following conclusions are made with regard to the steady-state error of a system with parabolic input:

- Type 0 system: $e_{ss} = \infty$
- Type 1 system: $e_{ss} = \infty$
- Type 2 system: $e_{ss} = R/k_a = constant$
- Type 3 or higher system: $e_{ss} = 0$

Example 4.5

Find the steady state errors of the following system

$$G(s) = \frac{k(s+3.15)}{s(s+1.5)(s+0.5)} \qquad \mathrm{H(s)} = 1$$

It is clear that this system is a type 1 system.
The steady-state errors are:

Step input	Step-error constant, $k_p = \infty$	$e_{ss}=R/(1+k_p) = 0$
Ramp input	Ramp-error constant, $k_v = 4.2k$	$e_{ss}=R/k_v = R/(4.2k)$
Parabolic input	Parabolic-error constant, $k_a = 0$	$e_{ss} = R/k_a = \infty$.

4.6.5 Steady-State Error for nonunity feedback system

For nonunity feedback control, we usually find the equivalent unity-feedback system, as shown in Fig. 4.13.

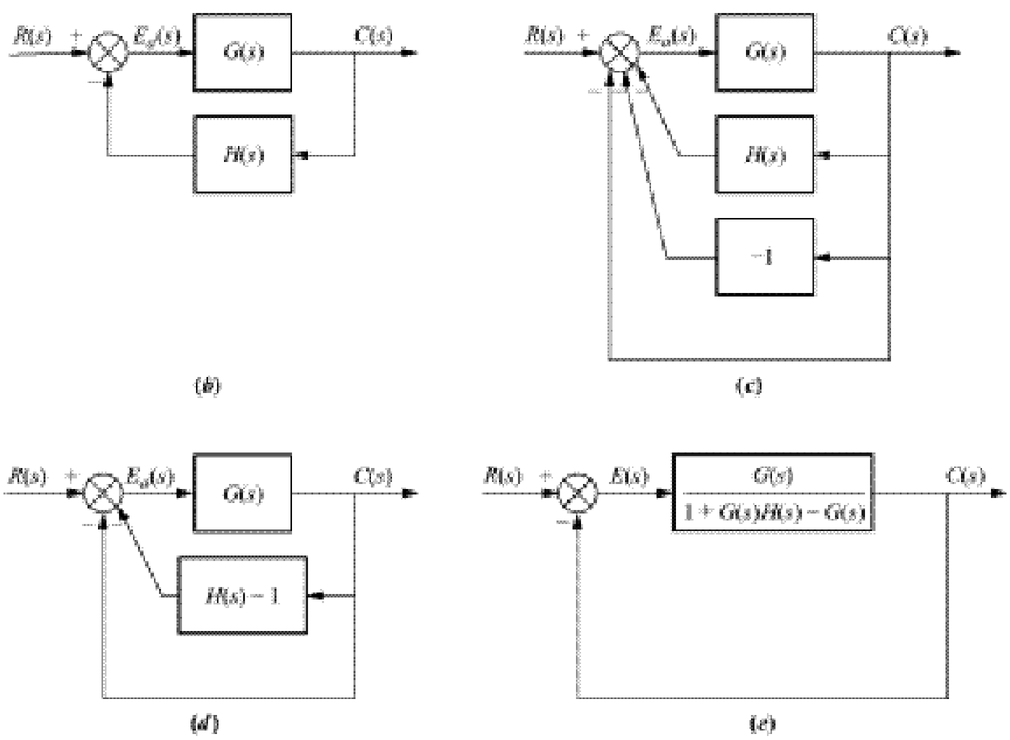

Figure 4.13 Forming an equivalent unity feedback for nonunity feedback system

We have to take into consideration that the above steps require the input and output in the same units.

The following example summarizes the concepts of steady-state error, system type, and the steady state errors.

Example 4.6

For the system shown in Figure 4.14, find the system type and the steady state error for the unit step function. Assume input and output units are the same.

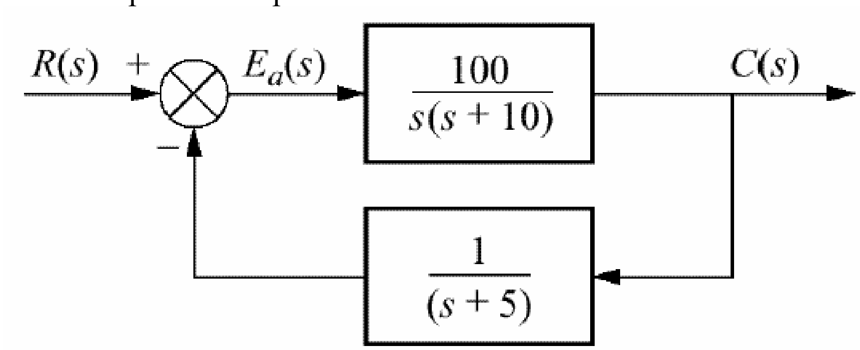

Figure 4.14 Nonunity feedback control system for Example 4.6

The first step in solving the problem is to convert the system of Fig. 4.14 into an equivalent unity feedback system. Using the equivalent forward transfer function of Figure 4.13 (e) along with

$G(s) = \dfrac{100}{s(s+10)}$ and $H(s) = \dfrac{1}{s+5}$,

we find $G_e(s) = \dfrac{G(s)}{1 + G(s)H(s) - G(s)} = \dfrac{100(s+5)}{s^3 + 15s^2 - 50s - 400}$.

Thus, the system is type 0,

and $k_p = \lim_{s \to 0} G_e(s) = \dfrac{100 \times 5}{-400} = -\dfrac{5}{4}$

The steady-state error is $e_{ss} = \dfrac{1}{1 + k_p} = -4$.